# On the creative role of axiomatics. The discovery of lattices by Schröder, Dedekind, Birkhoff, and others

*Forthcoming in Synthese, Special Issue: The Classical Model of Science II: The Axiomatic Method, the Order of Concepts and the Hierarcy of Sciences. Guest Editors: Arianna Betti, Willem R. de Jong and Marije Martijn*

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**Abstract.** Three different ways in which systems of axioms can contribute to the discovery of new notions are presented and they are illustrated by the various ways in which *lattices* have been introduced in mathematics by Schröder, Dedekind, Birkhoff, and others. These historical episodes reveal that the axiomatic method is not only a way of systematizing our knowledge, but that it can also be used as a fruitful tool for discovering and introducing new mathematical notions. Looked at it from this perspective, the *creative* aspect of axiomatics for mathematical practice is brought to the fore.

**Keywords:** Axiomatics, discovery, lattice theory, mathematical practice

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## 1. Introduction

### 1.1. On the creative role of axiomatics

It is quite common to regard axiomatic systems only as an aspect of the rigorous presentation of scientific or mathematical theories, or of the description of certain domains, but to deny them any role in the *creation* of new mathematics. This view is succinctly expressed, for example, in the following remarks by Felix Klein on the axiomatic treatment of group theory:

> The abstract [axiomatic] formulation is excellently suited for the elaboration of proofs, but it is clearly not suited for finding new ideas and methods, rather, it constitutes the end of a previous development. (Klein, 1926, 335)

Despite the fact that the importance of axiomatics for advancing mathematics had been clearly recognized and often emphasized by none other than Klein's successor in Göttingen, namely David Hilbert (Hilbert, 1918), Klein's view has remained very popular among mathematicians and even more so among philosophers.[1]

Traditional accounts of the use of axiomatics in science and mathematics often begin with a specific set of objects or a certain domain of being, say $D$, which an axiomatic system, say $S$, is intended to describe and characterize.[2] Understood in this way, axiomatization is the process of finding an adequate $S$ for a given $D$. However, Aristotle's brief remarks about the introduction of a *new* notion for what numbers, lines, solids, and times have in common, based on the similarity of certain proofs about them (*Analytica Posteriora* 1.5, 74a17–25),[3] suggest the following procedure: Take some domains $D_1$, $D_2$, $D_3$, etc. that are considered to be analogous in some respect and determine the corresponding axiomatic systems $S_1$, $S_2$, $S_3$, etc.; then, compare these systems and find a (sub-)system $S'$ that they have in common and introduce a *new* notion $D'$ as the domain of being for $S'$. Aristotle noticed that a scientific system $S'$ can be used in this way to suggest new notions, objects, or domains. Thus, axiomatization is not necessarily a one-way process from $D$ to $S$, but it can also lead one from $S'$ to $D'$. This insight presupposes neither the notion of a formal system, nor the possibility of multiple interpretations (although the latter would most likely be our way of expressing it). Since the domain $D'$ is more abstract (in the sense of having only a subset of the properties) than the domains $D_1$, $D_2$, $D_3$, etc., the natural setting for such introductions of new notions is mathematics, where the objects are inherently abstract. Indeed, the mathematical notion of magnitude was introduced later to express what the domains discussed by Aristotle have in common. With a conception of formal systems at hand, by which I mean systems with primitives that can be interpreted in different ways, and which emerged in the 19th century, a *second*, related way of introducing new domains became possible: Only certain aspects of a single domain $D$ are axiomatized by a system $S$, and then a new domain $D'$ is introduced that is completely determined by $S$. As a result, this new domain is more abstract than $D$ itself. Furthermore, an axiomatic system $S$ does not need to originate from a given domain $D$ at all, but it can also be obtained through modification from another system of axioms. For example, the first axiom systems for non-Euclidean geometry were obtained in this way from given systems of Euclidean geometry. Only after their consistency was established by interpreting the primitives in an Euclidean setting the new sets of objects, namely non-Euclidean points and lines, were introduced. This is a *third* way of introducing new domains.

Thus, we have identified three distinct ways in which axiomatics can contribute in an essential way to the introduction of new notions:

a) *By analogy:* Properties that different analogous domains have in common are expressed by a set of axioms, which, in turn, are taken as the definition of a new and more abstract notion. In other words, one begins with a prior conception of certain domains being similar and captures this similarity in terms of a common system of axioms.[4] These axioms are then understood as characterizing an abstract notion that is instantiated by the analogous domains.

b) *By abstraction:* Specific properties of a given domain are axiomatized and other domains are identified that also satisfy these axioms. In other words, one starts here with a particular mathematical domain and describes it axiomatically, thereby abstracting from all aspects that are deemed irrelevant. This axiomatic characterization then guides one to the discovery of other domains that satisfy the axioms and which are, on this basis, considered analogous.

c) *By modification:* A given axiomatic system is modified, by adding, deleting, or changing one or more axioms, and the resulting system is used as the definition of a new kind of domain.

For the new notions introduced in one of these ways to become accepted as genuine mathematics, in particular those that arise by modification, their underlying system of axioms must be considered to be 'interesting' in one way or another. We will see in the historical examples discussed below that a generally accepted sufficient reason for investigating an axiomatic system with genuine mathematical content is the fact that it describes a domain that had been investigated previously in its own right. This justifies considering the axioms as characterizing a new abstract notion and bars the introduction of notions based on a completely arbitrary set of axioms, since it guarantees a connection to the current body of mathematics.

In the remainder of this paper I will present and discuss how the notion of *lattice* has been introduced independently by Schröder, Dedekind, Birkhoff, and others, as examples of these three methods for introducing new domains on the basis of axiomatic systems, and I conclude that the axiomatic method is not only a way of systematizing our knowledge of specific domains, but that it can be — and has been — used as a fruitful tool for discovering and introducing new mathematical notions. Looked at it from this perspective and taking into account the role of axiomatics in modern mathematical practice, the *creative* aspect of axiomatics is brought to the fore.

## 1.2. The development of lattice theory

A *lattice* is an algebraic structure that can be defined in terms of two operations $\wedge$ (meet) and $\vee$ (join) that are commutative, associative, and satisfy the absorption laws $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$,[5] or, equivalently, in terms of a partial order relation on a domain in which the infimum and supremum of any two elements exist. This structure is instantiated in many different areas of mathematics, such as logic, set theory, algebra, geometry, functional analysis, and topology. Important for the development of lattice theory is their relation to another algebraic structure, namely Boolean algebras, which can be obtained by adding a complement relation to distributive lattices with 0 and 1 elements.

The history of the emergence of lattices and of the establishment of lattice theory as a well-respected and independent branch of mathematics has been investigated with great detail by Herbert Mehrtens in *Die Entstehung der Verbandstheorie* (The Genesis of Lattice Theory, 1979), on which I rely heavily in this presentation.[6] Mehrtens identifies three main sources for the notion of lattice: The set-theoretic grounding of mathematics, modern algebra, and the "axiomatic method" (Mehrtens, 1979, 292). With respect to the latter, he mentions two kinds of generalizations by which new notions can be introduced, which correspond to those referred to above as 'by analogy' and 'by abstraction' (Mehrtens, 1979, 197). Comparing the various developments that led to the independent introductions of the notion of lattice in the late 19th century and again in the 1930s, Mehrtens points out that all of these formations of a new notion resulted from generalizations, but none of them as part of a solution to some concrete problem.[7] Some particular episodes from the history of lattice theory that reveal the contributions of axiomatics are presented and discussed below.

## 2. Lattices obtained by modification of a system of axioms: Schröder's *logical calculus*

From early on in his career, Ernst Schröder (1841–1902) showed an interest in formal calculi and was well aware of the creative power of axiomatics, which he intended to exploit in his programme of *formal* or *absolute algebra*. He describes the general study of formal algebraic systems as proceeding in four stages in his textbook on arithmetic and algebra (1873): First, find all possible assumptions that could be used for defining an operation in a systematic and sufficient way, with consistency being the only restriction to be imposed on these assumptions. Second, investigate the consequences that can be derived from these assumptions. Third, try to find operations on number systems that are governed by the same laws, and fourth, determine what other meanings, e. g., geometric or physical, could be given to these operations (Schröder, 1873, 233 and 293–296). Four years later he presented an axiomatization of Boolean algebra as a calculus for classes and propositions, aiming at a minimal number of axioms and at a formal and rigorous presentation (Schröder, 1877). This axiomatization is based on two operations on classes, $+$ and $\cdot$ (understood as union and intersection, respectively), and it postulates that classes are closed under these operations, which are both associative, commutative, and idempotent; furthermore, that for any classes $a, b$ and $c$, $a = b$ implies $ac = bc$ and $a + c = b + c$, that they satisfy the distributivity law $a(b+c) = ab+ac$, that the universal class (denoted by '1') is the identity with respect to multiplication, and, finally, that for every class $a$ there exists a complement $a_1$, such that $aa_1 = 0$ (where '0' denotes the empty class) and $a + a_1 = 1$. As Mehrtens comments, Schröder's investigations are not as rigorous as announced (Mehrtens, 1979, 35–36); what is missing for a complete system of axioms for Boolean algebra are the (implicit) existence requirements for 0 and 1, and the condition that $0 \neq 1$. Nonetheless, Schröder is able to deduce from these axioms known theorems for Boolean algebra, including the absorption laws, and one distributivity law, $a+bc = (a+b)(a+c)$, while the dual one (i. e., $a(b+c) = ab+ac$) is taken as an axiom. However, an axiomatic characterization of lattices cannot be obtained directly by simply removing one or more axioms from this system. For this, a rearrangement of the system was necessary, which resulted a few years later after a brief exchange with C. S. Peirce.

Peirce had also investigated Boolean algebras and in his article "On the algebra of logic" he claimed that the distributivity laws are "easily proved

[...], but the proof is too tedious to give" (1880, 33). This remark must have caught Schröder's attention, since he began to study the independence of the distributive axiom himself. In the course of these investigations he split the axiom into the two inequalities $a(b+c) \Subset ab+ac$, and $ab+ac \Subset a(b+c)$, using the symbol $\Subset$ for subsumption, and he was able to show that the latter was provable from the remaining axioms, while the former was not. Schröder refers to Peirce's claim regarding the provability of the distributive laws in the first volume of his *Vorlesungen über die Algebra der Logik* (Lectures on the Algebra of Logic), published in 1890, noting that "This was a point that needed correction," and adding that one of the distributive inequalities was indeed easy and straightforward to prove.

> By no means, however, was I able to find a proof for the other part of the theorem. Instead, I was successful in showing its unprovability [...]. A correspondence with Mr. *Peirce* on this matter clarified the issue, since he also had become aware of his mistake. (Schröder, 1905, I, 290–291)[8]

To show the independence of the distributive law from the other axioms, Schröder employed what he called the method of proof "by exemplification" (Schröder, 1905, I, 286), which involves the now familiar presentation of a model in which the independent axiom is false, but the remaining ones are true. The quest for such a model led him to reconsider his earlier work on absolute algebra, where he found a suitable one in his *logical calculus with groups* ("logischer Kalkul mit Gruppen"). The results of these developments are presented in Schröder's lectures on the algebra of logic. In the first volume of these lectures he introduces an *identical calculus with subsets of a domain* ("identischer Kalkul mit Gebieten einer Mannigfaltigkeit") that is based on the primitive order relation $\Subset$. The basic assumptions for this calculus are:[9]

Principle I. $a \Subset a$.

Principle II. If $a \Subset b$ and $b \Subset c$, then $a \Subset c$.

Def. (1). If $a \Subset b$ and $b \Subset a$, then $a = b$.

Def. ($2_\times$). 0 is that subset for which $0 \Subset a$, for every subset $a$ of the domain.

Def. ($2_+$). 1 is that subset for which $a \Subset 1$, for every subset $a$ of the domain.

Def. ($3_\times$). If $c \Subset a$ and $c \Subset b$, then we say that $c \Subset ab$.

Def. ($3_+$). If $a \Subset c$ and $b \Subset c$, then we say that $a + b \Subset c$.

Postulate ($1_\times$). 0 is added as the *empty subset*.

Postulate ($1_+$). 1 is the entire domain.

Postulate ($2_\times$). $ab$ is that subset that is common to $a$ and $b$.

Postulate ($2_+$). $a + b$ is that subset that is formed by $a$ together with $b$.

Principle $\mathrm{III}_\times$. If $bc \Subset 0$ (and thus $bc = 0$), then $a(b + c) \Subset ab + ac$.

Def. (6). The *negation* of a subset $a$ is a subset $a_1$, such that $aa_1 \Subset 0$ and $1 \Subset a + a_1$ holds.

Postulate (3). For every subset $a$ there is at least one subset $a_1$, which can be obtained by omitting $a$ from the entire domain.

With minor changes (mainly of Definition $3_\times$ and Principle $\mathrm{III}_\times$) this system was later presented by Huntington as an axiomatization of Boolean algebra (1904). Schröder mentions six different areas of application for this calculus and points out that the conditions listed above Principle $\mathrm{III}_\times$ constitute a separate area of application, namely the *logical calculus with groups*, which is an instance of the modern notion of a lattice with zero and one elements. Schröder devotes three appendices to his lectures to a discussion of the logical calculus and he uses *algorithms*, which he had studied extensively in earlier publications, to exhibit a model for it. For Schröder, a *group* is simply a system that is closed under an operation and an *algorithm* is a group of formulas of a particular syntactic form. Of the form in question there are 990 different formulas, which constitute the universe $U$, and Schröder defines product and sum on algorithms, as well as the zero and one algorithm. While arbitrary subsets of $U$ together with union and intersection form a Boolean algebra, Schröder shows that the distributive law (Principle $\mathrm{III}_\times$) is not satisfied by the class of algorithms with operations suitably defined, while the laws from Principle I to Postulate $2_+$ are. Thus, he concludes that the notion determined by these postulates is of genuine mathematical interest and that the distributive law (and its dual, which can be proved from it) is independent from the other axioms. As Mehrtens emphasizes, it is the fact that this model has real content, i. e., that algorithms were studied before in their own right, which makes this independence result so important. Indeed, this is the only case in which Schröder proves the independence of an axiom, from which Mehrtens concludes that "this is not just an axiomatic technique, but the demarcation of two structures" (Mehrtens, 1979, 49–50).

The developments after the publication of the first volume of Schröder's lectures support the claim that an axiomatic definition of an abstract notion guides the discovery of other instances, since shortly afterwards other models were suggested for proving the independence of the distributivity axioms: Classes of natural numbers that are closed under addition (Lüroth, 1891), ideal contents of concepts (Voigt, 1892), and Euclidean and projective geometry (Korselt, 1894). These are discussed by Schröder in the second volume (1905) of his lectures on algebra (Schröder, 1905, II, 401–423).[10]

To summarize, the emergence of the notion of lattice in Schröder's work shows how an axiomatic characterization of a new mathematical notion can have its origin in a previous axiomatization, from which only a subset of the original axioms is considered. The particular axiom that led to this subset was brought to Schröder's attention by Peirce's investigations regarding its independence from the other axioms, and Schröder's own previous studies suggested to him a particular model for the remaining axioms, which was of independent interest.[11] This model justified him to regard the remaining axioms as determining a new notion, of which other mathematically interesting instances were subsequently found.

### 3. Lattices as abstractions: Dedekind's *Dualgruppen*

Richard Dedekind (1831–1916) was highly influential in developing the modern abstract style of mathematics and many of his results and techniques have become standard: He introduced such fundamental algebraic notions as field, module, and ideal, he formulated an axiomatic characterization of the natural numbers, and he gave the construction of a continuous domain in terms of cuts of rational numbers. What is perhaps less well-known, is that he also developed — more or less as a by-product of his work on algebraic number theory and independently of Schröder — the notion of lattices.

In algebraic number theory Dedekind's general aim was to transfer notions and results pertaining to elementary number theory to more general domains of numbers. Such a programme had begun with Gauss's investigations of the whole complex numbers of the form $a + bi$ ($a, b \in \mathbb{Z}$), now called 'Gaussian integers.' Kummer had extended this approach to the cyclotomic integers, solving the difficulty that decomposition into prime factors is not always unique by the introduction of ideal numbers.[12] This background explains some of Dedekind's seemingly unusual — for the modern reader — choice of terminology in algebra, which was deliberately chosen to highlight the analogies to number theory.[13] He published his main contributions to algebraic number theory as Supplements to the second (1871), third (1879), and fourth (1894) editions of Dirichlet's *Vorlesungen über Zahlentheorie* (Lectures on Number Theory), which Dedekind also edited.[14] It is telling for the depth of his work that Emmy Noether had her students read all versions of these supplements (Dedekind, 1964, Introduction). In the following, the interplay between Dedekind's axiomatic approach[15] and the emergence of his notion of lattice, which he called *Dualgruppe*, is presented.

An important technique for the formation of mathematical notions, which Dedekind employed as early as 1857, is to consider the set of objects that have a certain property as a single entity. For example, Dedekind considered congruent numbers and those numbers that are divisible by one of Kummer's ideal numbers as single mathematical objects. Similar considerations in his work on modules led him to the notion of lattice. When Dedekind first introduced the notions of fields, modules, and ideals in 1871, the operations on these entities were not part of the definitions themselves. Rather, they were induced from the underlying domain of numbers. Thus, a *module* was defined simply as a system of real or complex numbers that is closed under addition and subtraction (Dedekind, 1932, III, 242). Relations between modules, like being a divisor and multiple, as well as the notion of greatest common divisor (*gcd*) and least common multiple (*lcm*), were defined in terms of the underlying domain, but no symbols were introduced for these operations and Dedekind did not investigate them further. Only six years later, and with some hesitation, Dedekind introduced symbols for multiple ($>$), divisor ($<$), *lcm* ($+$), and *gcd* ($-$) in (Dedekind, 1877b, 121). This allows him to concisely state the following theorems (without proof), for modules $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$:

$$(\mathfrak{a}+\mathfrak{b})-(\mathfrak{a}+\mathfrak{c}) = \mathfrak{a}+(\mathfrak{b}-(\mathfrak{a}+\mathfrak{c})), \quad \text{and}$$
$$(\mathfrak{a}-\mathfrak{b})+(\mathfrak{a}-\mathfrak{c}) = \mathfrak{a}-(\mathfrak{b}+(\mathfrak{a}-\mathfrak{c})).$$

These correspond to what are now called the 'modular laws' in the theory of lattices and they illustrate the need for the introduction of symbolic representations for *gcd* and *lcm* in order to express such general facts. Dedekind also noted that these "characteristic theorems" display a dualism that holds throughout for the notions of *gcd* and *lcm*. That is, any true formula expressed in terms of $+$ and $-$ can be transformed into another true formula by switching these symbols. Mehrtens also mentions notes from Dedekind's *Nachlaß* entitled "Über den Dualismus in den Gesetzen der Zahlen Moduln" (On the dualism in the laws of number modules)[16], which reveals his interest for this particular phenomenon. In the 1894 version of the Supplements to Dirichlet's lectures Dedekind speaks of "a peculiar [*eigentümlicher*] dualism" (Dedekind, 1932, III, 66) between *gcd* and *lcm*. He introduces these operations separately and shows their fundamental properties, i. e., commutativity, associativity, and idempotency, for modules $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ (Dedekind, 1932, III, 63 and 65):

$$\begin{aligned} \mathfrak{a}+\mathfrak{b} &= \mathfrak{b}+\mathfrak{a}, & \mathfrak{a}-\mathfrak{b} &= \mathfrak{b}-\mathfrak{a},\\ (\mathfrak{a}+\mathfrak{b})+\mathfrak{c} &= \mathfrak{a}+(\mathfrak{b}+\mathfrak{c}), & (\mathfrak{a}-\mathfrak{b})-\mathfrak{c} &= \mathfrak{a}-(\mathfrak{b}-\mathfrak{c}),\\ \mathfrak{a}+\mathfrak{a} &= \mathfrak{a}, \quad \text{and} & \mathfrak{a}-\mathfrak{a} &= \mathfrak{a}. \end{aligned}$$

Together with the modular laws, which are now proved, the symmetry of these two operations becomes quite apparent. In a footnote to these considerations Dedekind introduces the notion of a *Modulgruppe:*

> If one repeatedly generates modules by forming greatest common divisors and least common multiples, beginning from three arbitrary modules, one obtains a finite *Modulgruppe*, which consists in general of 28 different modules. The peculiar laws of this group, which contains the modules $\mathfrak{a}\pm\mathfrak{b}$ if it contains the modules $\mathfrak{a}$ and $\mathfrak{b}$, shall be discussed elsewhere [cf. XXX]. (Dedekind, 1932, III, §169, 66–67)[17]

The *Modulgruppe* is also mentioned later in the text in a footnote, where ideals are introduced as a special kind of modules. Dedekind remarks that the group in question is reduced to 18 elements, if its elements are ideals rather than modules, which indicates that he had already studied in some detail the structures induced by the *gcd* and *lcm*.

The notion of lattice is finally introduced under the name *Dualgruppe* in (Dedekind, 1897) and studied further in (Dedekind, 1900). In the first of these articles, Dedekind studies systems of numbers in terms of their *gcd*s. This is done in the most general way possible, he explains, and is to be extended to domains which do not allow for decomposition into prime factors. Dedekind begins by investigating systems of three and four numbers, then systems consisting of $n$ general elements, called *combinations.* For these he formulates six fundamental laws for the operations of $-$ (the combination common to two given ones) and $+$ (the combination that contains two given ones), referred to as "laws $A$":

$$\begin{array}{rclrcl} \alpha+\beta & = & \beta+\alpha, & \alpha-\beta & = & \beta-\alpha, \\ (\alpha+\beta)+\gamma & = & \alpha+(\beta+\gamma), & (\alpha-\beta)-\gamma & = & \alpha-(\beta-\gamma), \\ \alpha+(\alpha-\beta) & = & \alpha, & \alpha-(\alpha+\beta) & = & \alpha. \end{array}$$

Thus, Dedekind identifies commutativity, associativity, and the absorption laws, and he also notes that the idempotent laws $\alpha+\alpha=\alpha$ and $\alpha-\alpha=\alpha$ follow from them, but that the distributive laws

$$\begin{array}{rcll} (\alpha-\beta)+(\alpha-\gamma) & = & \alpha-(\beta+\gamma) & \text{and} \\ (\alpha+\beta)-(\alpha+\gamma) & = & \alpha+(\beta-\gamma), & \end{array}$$

although true for the combinations considered, are not deducible from the laws $A$.

Since Dedekind's combinations are sets of elements, the operations of $+$ and $-$ can also be interpreted as union and intersection. Seen in this way, Dedekind remarks, many of his theorems about combinations correspond to theorems proved in Schröder's lectures on the algebra of logic, and he attributes "particular importance" to the fact that Schröder showed the independence of the distributive laws from the system of laws $A$. In fact, he remarks that he had dealt with these questions for many years himself and that he had also arrived at this result "not without great effort" (Dedekind, 1897, 113). In the subsequent paragraph Dedekind gives the following definition for the notion of *Dualgruppe:*

> A system $\mathfrak{A}$ of things $\alpha, \beta, \gamma \ldots$ is called a *Dualgruppe*, if there are two operations $\pm$, such that they create from two things $\alpha, \beta$ two things $\alpha \pm \beta$ that are also in $\mathfrak{A}$ and that satisfy the conditions $A$. (Dedekind, 1897, 113)

To be sure, *Dualgruppen* are not groups in the modern sense, but lattices. And although Dedekind himself had studied groups in the 1850s and an axiomatic characterization of groups had been published by Dyck in 1882,[18] it appears that this term was not always used in this sense by mathematicians who were not deeply involved with the theory of groups. For them, including Dedekind and Schröder, a group was simply a set of elements that is closed under certain operations.[19]

Immediately after the above definition, Dedekind continues: “In order to show how multifarious the domains are to which this concept can be applied, I mention the following examples” (Dedekind, 1897, 113), and in addition to the model provided by Schröder, he describes five other models, namely modules, ideals, the subgroups of a group, fields, and points of an $n$-dimensional space. Referring to this list of examples, Birkhoff, who takes historical accuracy very seriously, remarks that “[t]he abundance of lattices in mathematics was apparently not realized before Dedekind” (Birkhoff, 1940, 16). Thus, in this case the axiomatic definition of an abstract structure goes hand in hand with the observation that other mathematical domains also satisfy the axioms.

Dedekind not only gives an axiomatization of lattices, but he also develops the theory further. As already noted, the distributive laws do not hold in general for *Dualgruppen*, but they do for the important models from logic and ideal theory. This leads Dedekind to introduce the subspecies of *Dualgruppe vom Idealtypus*, i. e., distributive lattices. The lattice of modules, for which the modular laws hold, is called *Dualgruppe vom Modultypus*, accordingly. By exhibiting suitable models, Dedekind is able to show that these two notions do not coincide, i. e., that the distributive and modular laws are independent. In “Über die von drei Moduln erzeugte Dualgruppe” (On the *Dualgruppe* generated by three modules, 1900) Dedekind investigates — in modern terms — the free modular lattice with three generators; he also determines the structure of the lattice generated by three ideals (i. e., the free distributive lattice with three generators), and he further investigates modular lattices, proving some fundamental theorems about them.[20]

Mehrtens notes that in the references to Schröder’s lectures, Dedekind does not mention that his own notion of *Dualgruppe* coincides with Schröder’s notion of *logical calculus*, and speculates that at this point Dedekind might not have realized that the same abstract structure underlies his and Schröder’s investigations, but only that certain similar relations, which are expressed by the underlying axioms, hold between statements concerning logic and his ideals and modules (Mehrtens, 1979, 97). Thus, if Mehrtens’s analysis is correct, it were the axioms and theorems that brought out the analogy between Schröder’s and Dedekind’s work, when it was not yet possible for Dedekind to *grasp*, nor *see* the abstract structure that is instantiated. Three years later, however, Dedekind explicitly draws the connection between his *Modulgruppen* and Schröder’s *identical calculus*, and the correspondence between *Dualgruppen* and the *logical calculus* (Dedekind, 1900, 252, footnote).

Thus, by 1900 Dedekind had published an axiomatic characterization of lattices, discussed the main examples, and proved the fundamental theorems concerning modular lattices; but he did not present this work as the programmatic beginning of a new and important theory. Although abstract, Dedekind’s notion of *Dualgruppe* was intimately tied to that of modules, which he had hoped would play a fundamental role in algebraic number theory.

However, the subsequent development did not follow his lead, as can be seen from the fact that they do not occur in Weber's textbook on algebra (1895–1896) and that they are mentioned only very briefly in Hilbert's influential *Zahlbericht* of 1897.[21] In contrast to Schröder, whose starting point for the development of the notion of lattice was an axiomatization of Boolean algebras, Dedekind's investigations began as the study of particular instances. The structure generated by the operations of *gcd* and *lcm* on modules gradually emerged in these investigations, and the duality of the laws governing these operations sparked Dedekind's interest. He tried to give a minimal axiomatic characterization, and in the study of the dependencies between axioms and theorems it was especially the independence of the distributive law that caught his attention. Finally, the publication of Schröder's lectures provided him with a new example of an instance of this notion, which motivated Dedekind to publish his own investigations on these matters,[22] and, as soon as the axiom system was formulated, Dedekind noticed a number of other instances.

## 4. Lattices in the 1930s: analogies, modifications, and abstractions

Schröder's and Dedekind's notions of *logical calculus* and *Dualgruppe* were not taken up immediately by their contemporaries. Mathematical practice, however, changed substantially between the turn of the century and the 1930s. In particular, axiomatics developed into a general technique and the use of set-theoretic reasoning became commonly accepted. Moreover, by 1930 many algebraic structures had been studied extensively, and generalizations and abstractions were no longer frowned upon as they often had been earlier.[23] In the wake of these changes the notion of lattice was rediscovered independently by several authors around the same time. Mehrtens describes the years between 1930 and 1940 as the formation period of lattice theory, after which it had become an established mathematical theory. It is characteristic for this formation period that mathematicians who studied lattices still had to justify their interest in this notion to their peers. One of the pioneers of lattice theory, Garrett Birkhoff, reports:

> I recall being dashed when my father asked me what, specifically, could I prove using lattices that could not be proved without them! My lattice-theoretic arguments seemed to me so much more beautiful, and to bring out so much more vividly the essence of the considerations involved, that they were obviously the 'right' proofs to use. (Birkhoff, 1970, 6; quoted from Mehrtens, 1979, 176)[24]

A justification for a new notion was usually given in terms of their wide range of applicability or, if possible, their usefulness for solving problems. In the following I shall briefly present how the notion of lattice emerged by analogy in the work of Menger and Bennett, by modification in Klein, and by abstraction in Ore and Birkhoff.

### 4.1. Menger's unification of geometry

That a projective geometry can be seen as an instance of a lattice had been noted by several authors (e. g., Korselt 1894), but only for Karl Menger (1902–1985) this was the main motivation for introducing his notion of *Feld*, i. e., a lattice with 0 and 1 elements. He was surprised by the fact that projective and affine geometry,[25] although analogous in many respects, were not presented by similar axiomatizations, and asked: "Since projective and affine geometry have so much in common, why not base them on two sets of assumptions that have much in common?" (Menger, 1940, 43; translated from Mehrtens, 1979, 132).

Menger's interest in geometry was sparked by a coincidence. When he was assigned to teach projective geometry at the beginning of his career as professor in Vienna in 1927, he could not find a satisfying foundation of it in terms of union and intersection, and so he decided to work one out by himself.[26] This resulted in a few remarks on a new axiomatization of projective geometry (1928), and Menger continued these investigations together with his students, but they did not arouse much interest outside of their circle.[27] In the course of several years this system of axioms was studied in depth and more and more refined, and a summary of these efforts was published as "New foundations for projective and affine geometry" — subtitled "Algebra of geometry" — in 1936. Menger explicitly motivates his axiomatization of projective geometry, which is based on a single domain of entities (the linear parts of a space) and two operations of union and intersection, by the "far-reaching analogy" with abstract algebra and the algebra of logic that is thereby obtained (Menger, 1936, 456). He recalls:

> The algebra of numbers has been developed from postulates about adding and multiplying numbers; the algebra of classes from postulates about joining and intersecting classes. This suggested a foundation of geometry on postulates about joining and intersecting flats, and the name 'algebra of geometry' for the theory developed. (Menger, 1940, 45; quoted from Mehrtens, 1979, 132)

In addition, his treatment of geometry differs from the traditional ones in two other respects, namely that the geometry has an arbitrary finite number of dimensions from the start, and that affine and projective geometry are developed together as much as possible. Menger explains:

> We first develop [ . . . ] consequences of a system of axioms valid in both affine and in projective spaces. [ . . . ] From this system of axioms common to both geometries we pass to either of them. By adjoining the missing dual of one axiom we obtain a completely self-dual system from which all of projective geometry can be deduced. By adding the Euclidean parallel axiom we obtain the theory of affine spaces. (Menger, 1936, 457)

The possibility of developing great parts of both theories together "could hardly have been foreseen," Menger remarks, and he also claims that this has advantages "from the pedagogic point of view" (Menger, 1936, 457). In the concluding paragraph he suggests further investigations based on his axiomatization:

> By varying slightly some of the axioms of our system, new geometry systems might be obtained. [Footnote: This matter is evidently related to the question of the independence of our axioms, which is not considered in this paper.] Particularly promising in this respect is a variation of Axiom ·6 which, as we have seen constitutes the single difference between projective and affine geometry. (Menger, 1936, 481)

Thus, after having introduced his axiomatization based on a perceived analogy between affine and projective geometry, Menger very clearly expresses here how modifications of it can lead to new theories and he identifies one axiom from his system which looks "particularly promising." In other words, he employs axiomatics not just for unifying different theories, teaching, and consolidating previous results, but also as a vehicle for further investigations.

### 4.2. Bennett's explication of a commonality of axiom systems

Although Albert Bennett's (1888–1971) paper on lattices (1930) remained fairly isolated and consists in not much more than the definition of lattices, it is worth discussing at this point for two reasons. First, it was presented at the time when other similar formulations emerged, thus indicating that the rediscovery of the lattice structure was in the air. Second, he explicitly motivates the introduction of this notion by pointing out that it captures what is common to various previously studied notions. Thus, like Menger's, also Bennett's axiomatization is intended to clarify an analogy between previously given mathematical notions.

Bennett begins his paper by noting that the notion of serial order (i. e., total order) and the calculus of classes have received a fair amount of attention from the "postulationists," in particular by Huntington in (1904) and (1917). He continues:

> The two subjects differ considerably but both may be developed by use of a common symbol, $<$, of order relation. Some other important systems differing from both show also an essentially analogous use of a symbol of dyadic order relation [ . . . ]. It appears therefore worth noting that a body of common relations found in these various basic mathematical studies has hitherto escaped a common formulation. (Bennett, 1930, 418)

Obviously unaware of the earlier work by Peirce and Schröder, Bennett presents an axiomatization of *semi-serial order* (partially ordered sets, with suprema and infima). The axiom system is taken mostly from Huntington, but "by the omission of certain postulates there given but here extraneous and by introducing VIII an essentially new system of more extensive application is obtained" (Bennett, 1930, 419).[28] Thus, Bennett's aim is to characterize a perceived analogy axiomatically, and this analogy is based on previous axiomatizations. This is similar to the case of Aristotle mentioned in the introduction. For this reason it is appropriate to refer to his introduction of lattices as 'by analogy,' since he would not have pursued this axiomatization without being motivated by the analogy he saw between serial orders and the calculus of classes.

After presenting his axioms Bennett shows how to define the operations $\vee$ and $\wedge$ from the order relation and he deduces some basic theorems. The last two of this five page paper is devoted to a list of 12 mathematical domains that satisfy his system of axioms. He mentions the natural numbers together with $\omega$ and the non-negative real numbers with $-1$ and $+\infty$, both domains with the usual order relation; the non-negative rational integers with *gcd* and *lcm*; the subclasses of a given class with logical product and sum; the linear projective subspaces of a given space of $n$ dimensions. In this context Bennett notices a new connection between logic and geometry, namely that the algebra of logic applied to a class of $n + 1$ elements is a special case of Veblen and Young's theory of finite geometry of $n$ dimensions with $p + 1$ points on a line, where $p = 1$, and he remarks that "[t]his relationship is however left unnoted by Veblen and Young, and by Huntington" (Bennett, 1930, 422). As further examples Bennett lists the set of closed intervals on a line, the set of all convex regions in a plane, the set of all submodules of a given module, the class of all linear subsets of a given linear set, the class of all regions in the plane each of which is bounded by a circle, the system of subgroups of a given group, and the set of idempotent elements in certain algebras of finite basis, all with appropriate operations.

In sum, this short paper is a great example of the use of axiomatics for capturing what is common to two given theories and of the fact that other models can be found with ease once such an axiomatization is formulated.

### 4.3. Klein's generalization of algebraic structures

In contrast to Bennett, who apparently wrote only a single paper on lattices, Fritz Klein (1892–1961) published over a dozen of articles on this topic between the years 1929 and 1939. Initially, influenced by the work of Schröder and his own investigations of logic, he became interested in abstract operations, i. e., where the nature of the elements that are operated upon can be disregarded (Klein, 1931, 398). In particular, he found it curious that the distributive laws of logical sum and product are symmetric, while those for arithmetical addition and multiplication are not (1929). Thus, a negative analogy caught his attention. Following up on this observation he was led to the axioms for a distributive lattice, calling the notion an *"A-Menge"* (1931), and a year later he introduced the general notion of lattice under the current German term *"Verband"* (Klein, 1932, 117). In this context he also gave examples from number theory, which he had apparently learned in the meantime from Dedekind's works. More references to other models appear in the later publications, and Klein seems to have been encouraged in his pursuits by realizing that other mathematicians had also independently found interest in the notion of lattice. Like Bennett, the focus of Klein's research lies in pure axiomatics, and the applications merely serve to provide a justification. They do not appear to guide his investigations in any particular way. According to Mehrtens, Klein's studies are detailed but elementary; e. g., he does not discuss the modular laws at all, but his work is evidence for a change towards a mathematical practice that emphasizes abstract axiomatic approaches (Mehrtens, 1979, 174–175).

### 4.4. ORE'S PROGRAMME OF STRUCTURAL INVESTIGATIONS

A much more influential contribution to lattice theory than that provided by the three mathematicians discussed above is the work of Oystein Ore (1899–1968). After obtaining his doctorate in 1924 under Skolem, Ore worked chiefly on algebraic number theory until the 1930s, focusing in particular on field and ideal theory. Together with Noether and Fricke he edited Dedekind's collected works (1930–32), and by this time his interest shifted to polynomials in non-commutative rings. His general aim was the transfer of decomposition theorems from algebraic number theory to non-commutative domains. These investigations were extended later to include the Jordan-Hölder theorem and group theory, where his goal became to "base the theory [of groups] as far as possible directly upon the properties of subgroups and eliminate the elements" (Ore, 1937, 149; translated from Mehrtens, 1979, 211). A general discussion of Ore's contribution to the structural image of algebra can be found in (Corry, 1996, Ch. 6, 263–292). I shall focus here on the emergence of the notion of lattice in his works.

Ore was interested in general structural properties of algebraic systems, and he found in lattices, which he called *structures*, a very fruitful tool for his investigations. Birkhoff and Mac Lane suggest (in letters to Mehrtens) that Ore developed this notion independently, despite the fact that his first publication on this subject was after Birkhoff's, whom he mentions. His "On the foundation of abstract algebra. I" was for many mathematicians the first time they heard about the notion of lattice and it was one of the most often quoted papers on this subject until 1940.[29] Ore begins this paper by rejecting the search for a general notion that encompasses all algebraic structures, but suggesting instead a different approach to their unification and study, namely through the investigation of the systems of relations between their sub-domains in terms of a new notion:

> For all these systems there are defined the *two operations* of *union* and *cross-cut* satisfying the ordinary axioms. This leads naturally to the introduction of new systems, which we shall call *structures*, having these two operations. (Ore, 1935, 406)

He notes that, on the one hand, this more abstract approach results in a loss of available mathematical machinery (e. g., residue systems and cosets), but that, on the other hand, “a great deal of simplification and also many new results” are gained by this move (Ore, 1935, 407). The importance of new results is also emphasized in a later paper, where he remarks, in connection with the possibility of presenting known results from different areas as following from a common unifying notion:

> It is of course quite interesting to examine to what extent this is possible, but the real usefulness of the idea appears through the various new results to which it leads. (Ore, 1938, 801, quoted from Corry, 1996, 274)

Previous mathematicians had introduced the notion of lattice either in terms of a partial order relation and then defined the operations of meet and join from it (e. g., Bennett), or vice versa (e. g., Menger and Klein). Ore shows that these axiomatizations are in fact equivalent (Ore, 1935, 409).[30] I have pointed out in connection with Bennett, that until 1930 the notion of order had almost exclusively been understood as linear (total) order. As such it had been investigated axiomatically by Huntington and Veblen. Hausdorff had introduced the notion of *partially ordered set* in 1914, but omitted it from the revised second edition of his textbook in 1927, since it was of no further importance for his work. It is chiefly with Ore’s axiomatic presentation of a “*partly ordered set*” and the relevance of partial orderings in connection with lattices that this notion became more prominent in mathematics.[31] At this point an interesting observation can be made regarding the difference between how mathematics is perceived and presented in retrospect and how it actually develops. Birkhoff tells us the following story:

> It is often said that mathematics is a language. If so, group theory provides the proper vocabulary for discussing *symmetry*. In the same way, lattice theory provides the proper vocabulary for discussing *order*, and especially systems which are in any sense hierarchies. (Birkhoff, 1938, 793; translated from Mehrtens, 1979, 314)

As we have seen, this logic internal to mathematics does not reflect the historical development of the theory, in which the study of orderings played only a marginal role.

In a brief passage Ore gives us some insight into his systematic investigations of possible systems of axioms. After having introduced the modular and distributive axioms (referred to as “Dedekind axiom” and “Arithmetic axiom”), Ore notes that “[t]he most common algebraic systems satisfy axioms less restrictive than the arithmetic axiom and more restrictive than the Dedekind axiom" (Ore, 1935, 415); he presents a method for obtaining axioms of intermediate strength, by imposing identities on the 18 elements that are generated by union and intersection of three given elements, i. e., on the free distributive lattice generated by three elements.[32] Thus, the idea is to systematically construct possible models and then to formulate axioms that allow to distinguish between these models. Unfortunately, in the case at hand this method does not succeed, as Ore explains:

> A discussion of all possibilities shows however that, aside from trivial cases, all conditions thus obtained are equivalent to the arithmetic axiom. This then proves that the arithmetic axiom is the only stronger axiom containing only three arbitrary elements $A, B, C$. To obtain other axioms stronger than the Dedekind axiom it is necessary to consider the general Dedekind structure generated by four or more elements. (Ore, 1935, 415)

However, Birkhoff (1933) had shown that such structures are in general infinite, by which "the quest for such special axioms is considerably complicated" (Ore, 1935, 415).

To conclude, Ore's notion of lattice is intended as a tool for generalizing and investigating algebraic structures. Thus, his way of arriving at lattices is, like Dedekind's, by abstraction. This becomes clear from the fact that his main justification for the new notion is that it allows to recapture important algebraic decomposition theorems and that he does not mention any models from areas of mathematics other than algebra. As Mehrtens points out, this is not enough to form the basis for an independent theory (Mehrtens, 1979, 186). Such a basis was developed by Garrett Birkhoff.

### 4.5. Birkhoff's consolidation of lattice theory

When Garrett Birkhoff (1911–1996) was born, his father, G. D. Birkhoff, was one of the leading American mathematicians. Garrett received his B. A. in 1932, then went to England to study group theory, and soon thereafter published his first work of lattice theory (1933). As his main influences Birkhoff mentions the group theorist Remak and the algebraist van der Waerden.[33] The latter's 1930 book *Moderne Algebra*, based on lectures by Artin and Noether, was the first and highly influential, cohesive, and abstract presentation of algebraic structures. In the preface its aim is described as an introduction to a "whole world" of algebraic concepts, and the creative role of axiomatics is acknowledged:

> The recent expansion of algebra far beyond its former bounds is mainly due to the "abstract," "formal," or "axiomatic" school. This school has created a number of novel concepts, revealed hitherto unknown interrelations and led to far-reaching results, especially in the theories of *fields* and *ideals*, of *groups*, and *hypercomplex numbers*. (van der Waerden, 1930, quoted from the translation of the second edition, xi);

General notions, like those of structure-preserving mappings and equivalence relations, are introduced set-theoretically in the first chapter, and then they are applied in the study of groups, rings, fields, etc., which are introduced axiomatically. Just before the definitions of rings and fields, the general notion of a *system of double composition* is introduced as a system that is closed under the operations of addition and multiplication, which encompasses also lattices, but the latter are not mentioned (van der Waerden, 1930, 37). Through this book Birkhoff became acquainted with a variety of different algebraic structures, while Remak's work showed him the importance of the study of substructures.

Remak investigated unique decomposition of finite groups and the representation of finite groups as subgroups of direct products. Particular aspects of his work are the use of the structure of the normal subgroups of a group and the investigation of the subgroups of direct products with three factors (1932). Mehrtens speculates that Birkhoff may have begun his investigation of the subgroup generated from three normal subgroups by repeatedly forming direct products in connection with his studies of Remak's works. This structure corresponds to the free modular lattice generated by three elements and is the same that Dedekind had investigated more than three decades earlier. It plays a prominent role in Birkhoff's first paper on lattice theory (1933), where he repeats many of Dedekind's results, but also presents new material;[34] e. g., that a lattice generated in this way by four elements (called "free") is in general infinite and that every distributive lattice can be represented by a ring of sets.[35] Since it seems that Birkhoff's motivation for his axiomatization were particular instances of lattices, his introduction is one by abstraction, similar to that of Dedekind. Birkhoff also discusses modular and distributive lattices, and applications to group theory, ideal theory, and geometry. These applications are elaborated in later papers, where further ones are added, e. g., set theory, measure and probability theory, equivalence relations, and topology.

The work of Garrett Birkhoff was instrumental for establishing lattice theory as an independent and generally accepted mathematical theory. Not only did he introduce the now common English term "lattice" in 1933 and wrote the first monograph on this subject in 1940, but he also developed the theory in great depth and he was able to integrate the work of other mathematicians into one coherent whole. Bennett and Klein showed that many algebraic structures can be studied as lattices, Menger and Ore showed the relation of lattice theory to the foundations of geometry and to the decomposition theorems in algebra, but Birkhoff is the one who really emphasized the notion of lattice as being of central importance to many mathematical fields. In the opening lecture of the 1938 spring meeting of the American Mathematical Society he introduced lattice theory as a "vigorous and promising younger brother of group theory" and argued emphatically that some familiarity with it "is an essential preliminary to the full understanding of logic, set theory, probability, functional analysis, projective geometry, the decomposition theorems of abstract algebra, and many other branches of mathematics" (Birkhoff, 1938, 793; quoted from Mehrtens, 1979, 284).[36]

## 5. Conclusion

In all the investigations discussed in this paper, axiomatics has been a key *methodological* and *creative* tool for mathematical discovery. We have seen that the abstract notion that is today called 'lattice' was developed independently by Ernst Schröder and Richard Dedekind in the late 19th century. Schröder was led by considerations regarding the independence of the distributive axioms to a meaningful instance of a lattice, which in turn justified his isolation of a subset of the axioms of Boolean algebras. For Dedekind, the structures induced by the operations of *gcd* and *lcm* on modules were the instances of lattices that motivated his axiomatic characterization, and, once the notion was presented axiomatically, he quickly found further instances from many different areas of mathematics. However, their notions were not taken up by their contemporaries and thus lay dormant for the next decades, but by the 1930s a number of developments had taken place in mathematics that facilitated the development and spreading of abstract notions. These include, in particular, a general acceptance of set-theoretic and axiomatic reasoning, and of the study of abstract structures in their own right. In this context the notion of lattice reemerged in the quest for unifying notions in the independent work of younger mathematicians (Klein, Bennett, Menger, Ore, and Birkhoff).[37] Indeed, its unifying power is now, in retrospect, regarded as one of the most important virtues.[38] Within a decade this research had been consolidated and lattice theory had been established as an independent branch of mathematics, which is marked by the publication of the first textbook on lattice theory by Birkhoff (1940). These developments surrounding the emergence of the notion of lattice were intimately connected to the use of axiomatics. In particular, they illustrate the three different ways I have identified in the Introduction by which axiom systems can contribute to the introduction of new notions: *by modification*, *by abstraction*, and *by analogy*. Thus, the creative aspect of axiomatics is an essential ingredient of mathematical practice.

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## Notes

[1]For a more in-depth discussion of this point, see (Schlimm, 2006).

[2]See, for example, (de Jong and Betti, 2008).

[3]For a detailed discussion of this passage, see (Hasper, 2006).

[4]On the use of axioms to characterize analogies, see also (Schlimm, 2008b).

[5]In an axiomatic definition of lattices one can also use idempotency together with the equivalence $(a \wedge b = a) \leftrightarrow (a \vee b = a)$ instead of the absorption laws; see (Birkhoff, 1933) and (Klein, 1935).

[6]On the development of Boolean algebra, see also (Serfati, 2007).

[7]This observation contradicts the popular claim that the notions of modern mathematics arose by necessity in order to solve earlier problems. See also the quotation from Birkhoff at the beginning of Section 4, and (Schlimm, 2008a).

[8]See also (Peirce, 1885, 190) for Peirce's acknowledgement of Schröder's correction. For some later developments, see (Huntington, 1904, 300–301) for excerpts from a letter from Peirce to Huntington on this issue, and the discussion in (Peirce, 1966, III, 128) and (Mehrtens, 1979, 47–48).

[9]These can be found on pages 168, 170, 184, 188, 196, 212, 214, 293, 302, and 303 of volume I of (Schröder, 1905); see also (Mehrtens, 1979, 43–44).

[10]See also (Mehrtens, 1979, 59).

[11]Notice that many models constructed only for the purpose of showing the independence of certain axioms, e. g., in (Huntington, 1904), are of no further mathematical interest.

[12]The ring of cyclotomic integers is $\mathbb{Z}[\zeta_n]$, where $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a complex $n$th root of 1. The name derives from the fact that the points $\zeta_n, \zeta_n^2, \ldots, \zeta_n^n$ are equally spaced around the unit circle. See (Dedekind, 1877, 3–45) for a historical introduction by Stillwell.

[13]See, for example, (Dedekind, 1877, 64) and (Dedekind, 1932, III, 62).

[14]For a discussion of these works, see (Avigad, 2006).

[15]On Dedekind's axiomatic approach in his foundational work, see (Sieg and Schlimm, 2005).

[16](Cod. Ms. Dedekind XI, 1). This manuscript makes references to the second edition of the Supplements, thus being written before 1879, when the third edition was published.

[17]The reference XXX is to (Dedekind, 1900) and was added by the editors of the *Gesammelte Werke*. Note that Dedekind writes $\mathfrak{a} \pm \mathfrak{b}$ for '$\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} - \mathfrak{b}$.'

[18]See (Wussing, 1984).

[19]To avoid confusions arising from this terminology, I shall refer to Dedekind's notion simply as *Dualgruppe*, rather than translating it into English.

[20]See, for example, (Burris & Sankappanavar, 1981, 12–17).

[21]For a discussion of the lack of influence of both Schröder's and Dedekind's notions of lattices, see (Mehrtens, 1979, 123–126).

[22]Interestingly, also the publications of Dedekind's work on the foundations of mathematics were triggered by other publications on the same subject matter; see (Dedekind, 1872, 317) and (Dedekind, 1888, 335).

[23]See (Corry, 1996) and (Ferreirós, 1999) for detailed accounts of these developments.

[24]Birkhoff's contributions to lattice theory are discussed in Section 4.5, below.

[25]In *projective* geometry all lines intersect, points and lines are dual. *Affine* geometry is a theory common to Euclidean and several non-Euclidean geometries, which contains the notion of parallelism, but not that of a metric.

[26]An interesting historical parallel is Dedekind's interest in the foundations of analysis, which also resulted from his teaching duties (Dedekind, 1872, 315).

[27]See also (Mehrtens, 1979, 131).

[28]The 'VIII' refers to Bennett's list of axioms.

[29]Birkhoff's textbook *Lattice Theory* was published in 1940. His early publications on lattices were in lesser-known journals.

[30]Equivalence is meant here in the sense of mutual interpretability.

[31](Ore, 1935, 408). Ore refers to Hausdorff for the terminology; see also (Mehrtens, 1979, 187).

[32]This is Dedekind's *Dualgruppe vom Idealtypus*, i. e., a distributive lattice; see page 11, above.

[33]In a letter to Mehrtens (Mehrtens, 1979, 159).

[34]Birkhoff had been made aware of Dedekind's work by Ore, and he discusses the relation to his own work in (Birkhoff, 1934). He recalls: "Not knowing of Dedekind's previous work, I felt that my results partly justified my claims" (Birkhoff, 1970, 6; quoted from Mehrtens, 1979, 176). Later, he remarks: "I admired the style and the power of the master, and was glad that he had not anticipated more of my work" (Letter to Mehrtens, quoted from Mehrtens, 1979, 178). Mehrtens points out the striking similarities between Birkhoff's and Dedekind's approach; he remarks that Dedekind introduced his *Dualgruppen* when he was already retired, while Birkhoff developed his notion of lattice when he was only 22 years old: The knowledge that Dedekind had accumulated during his life had in the meantime become general knowledge in the community of algebraists (Mehrtens, 1979, 179).

[35]A *ring of sets* is a family of sets that are closed under finite union and intersection.

[36]Similar remarks can be found in the preface to the second edition of *Lattice theory* (Birkhoff, 1940; second ed., 1948, iii–iv).

[37]Of possible interest is also (Skolem, 1936), but I was not able to access this paper; see its review (Birkhoff, 1937).

[38]See, for example, (Birkhoff, 1970, 1).

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