Indecomposability of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ in Constructive Reverse Mathematics

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Abstract

It is shown that —over Bishop’s constructive mathematics— the indecomposability of $\mathbb{R}$ is equivalent to the statement that all functions from a complete metric space into a metric space are sequentially nondiscontinuous. Furthermore we prove that the indecomposability of $\mathbb{R} \setminus \{0\}$ is equivalent to the negation of the disjunctive version of Markov’s Principle. These results contribute to the programme of Constructive Reverse Mathematics.

Keywords: Indecomposability, Constructive Reverse Mathematics, Continuity Principles, Disjunctive Version of Markov’s Principle

1 Introduction

Bishop’s constructive mathematics (BISH) can be seen as lying in the intersection of classical mathematics, intuitionistic mathematics, and recursive mathematics. That means that we can characterise BISH as:

- classical mathematics without the Law of Excluded Middle and several Choice Principles;
- intuitionistic mathematics without principles of Bar-Induction, Continuity Principles and Kripke’s Scheme;
- recursive mathematics without Markov’s Principle and the Church-Turing Thesis.

Therefore BISH can be used as a base to study the logical strength of various non-constructive principles.

However, to make the—now important—distinction between pointwise continuous functions and uniformly continuous functions, we have to break with Bishop’s convention of studying only functions that are uniformly continuous on compact sets (and calling them continuous). Nor do we presuppose that all functions are pointwise continuous. Also unlike Bishop we will use $\neq$ for the negated equality, and $\#$ for the apartness relation.

This paper studies the indecomposability of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ within the programme of Constructive Reverse Mathematics.

Indecomposability — the property of a set that it cannot be effectively split — of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ has been investigated before. We give a brief overview of the relevant literature, starting with the indecomposability of $\mathbb{R}$:
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• In [3] a proof can be found of the indecomposability of a compact interval from which the indecomposability of $\mathbb{R}$ follows. This proof is carried out within intuitionistic mathematics and makes use not only of Continuity Principles, but also of the Fan Theorem\footnote{The Fan Theorem can be proved from Bar-Induction and Continuity Principles.} in the form of the Uniform Continuity Theorem. Apparently [5] Brouwer did something similar [4].

• The paper [6] shows the equivalence of the indecomposability of $\mathbb{R}$ and the statement that there are no discontinuous functions. However, this is done under the assumption of Kripke’s Scheme. Hence it leaves open the question to which (continuity) principle the indecomposability of $\mathbb{R}$ is equivalent over BISH.

• In [10] a proof of the indecomposability of $\mathbb{R}$ is implied (as an exercise), using the principle that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous, as the only non-constructive principle. However, as this is intended as part of a course in intuitionistic mathematics, and not as part of Constructive Reverse mathematics, no equivalent of the indecomposability of $\mathbb{R}$ is singled out.

One of the main contributions of this paper is to identify a Continuity Principle, $\text{CONT}_{\text{c00}}$, that is equivalent to the indecomposability of $\mathbb{R}$ over BISH. Because this continuity principle is both valid in intuitionistic mathematics and recursive mathematics (but not in classical mathematics), so is the indecomposability of $\mathbb{R}$.

That does not hold for the indecomposability of $\mathbb{R} \setminus \{0\}$. It has been remarked in [5] that Markov’s Principle is equivalent to $\mathbb{R} \setminus \{0\}$ being disconnected. We will show that the indecomposability of $\mathbb{R} \setminus \{0\}$ is equivalent to the negation of the disjunctive version of Markov’s Principle\footnote{I wish to thank the referee for pointing out this equivalence.} over BISH. This forms another main contribution.

2 The Indecomposability of $\mathbb{R}$

A set $X$ is indecomposable when every decidable subset of $X$ is either empty or $X$.

Recall that a mapping between metric spaces is said to be sequentially nondiscontinuous if for each sequence $(x_n)_{n \geq 0}$ converging to a limit $x$ and for each $\delta \in \mathbb{R}$ such that $\rho(f(x_n), f(x)) \geq \delta$ for all $n \in \mathbb{N}$, we may conclude that $\delta \leq 0$.

The Weak Limited Principle of Omniscience $\text{WLPO}$ is defined as follows:

$$\text{WLPO}: 0 \geq x \text{ or not } 0 \geq x \text{ for all } x \in \mathbb{R}.$$  

The Continuity Principle $\text{CONT}_{\text{c00}}$\footnote{I wish to thank the referee for pointing out this equivalence.} is the following statement:

$$\text{CONT}_{\text{c00}}: \text{Every mapping of a complete metric space into a metric space is sequentially nondiscontinuous.}$$

Using the fact that $\text{CONT}_{\text{c00}}$ is equivalent to $\neg \text{WLPO}$, we will show that the indecomposability of $\mathbb{R}$ is equivalent over BISH to $\text{CONT}_{\text{c00}}$.

THEOREM 2.1
The following statements are equivalent over BISH:

(1) $\text{CONT}_{\text{c00}}$.
(2) Every function from $\mathbb{R}$ to $\mathbb{N}$ is constant.
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(3) The set $\mathbb{R}$ is indecomposable.

**Proof.** We will prove that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$.

1 $\Rightarrow$ 2 Let $f$ be a function from $\mathbb{R}$ to $\mathbb{N}$. Suppose that there are $r, q \in \mathbb{R}$ such that $f(r) \neq f(q)$. Then define sequences $(r_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ of reals as follows:

$$r_0 := r \quad q_0 := q;$$

$$r_{n+1} := \begin{cases} \frac{r_n + q_n}{2} & \text{if } f(\frac{r_n + q_n}{2}) = f(r_n); \\
                          r_n & \text{else.} \end{cases}$$

$$q_{n+1} := \begin{cases} \frac{q_n + q_n}{2} & \text{if } f(\frac{r_n + q_n}{2}) = f(r_n); \\
                          q_n & \text{else.} \end{cases}$$

Now we see that

- $f(r_i) \neq f(q_i)$ for every $i \in \mathbb{N}$;
- for every $i \in \mathbb{N}$ we have $f(r_i) = f(r)$;
- for every $i \in \mathbb{N}$, $|r_i - q_i| \leq 2^{-i}|r - q|$.

Note that the two sequences converge to the same limit, say $z$. Suppose that $f(z) \neq f(r)$. Then for every $i \in \mathbb{N}$,

$$|f(r_i) - f(z)| = |f(r) - f(z)| \geq 1,$$

which is in contradiction with the sequential nondiscontinuity. Hence $f(z) = f(r)$ and $f(z) = f(r_i)$ for every $i \in \mathbb{N}$.

We remark now that for every $i \in \mathbb{N}$,

$$|f(q_i) - f(z)| = |f(r_i) - f(z)| + |f(q_i) - f(z)| \geq |f(r_i) - f(q_i)| \geq 1.$$

This also contradicts the sequential nondiscontinuity. Hence there are no two points with different function values.

2 $\Rightarrow$ 3 Suppose $A$ is a decidable subset of $\mathbb{R}$. Then $I_A$, the characteristic function of $A$, is well-defined and is constant. Hence $A = \emptyset$ or $A = \mathbb{R}$.

3 $\Rightarrow$ 1 Suppose that for each $x \in \mathbb{R}$ we can decide whether $0 \geq x$ or not $0 \geq x$. Note that we can then also decide for every $x \in \mathbb{R}$ whether $0 = |x|$ or not $0 = |x|$. That means that the set $\{0\}$ is decidable and hence it gives us a non-trivial decomposition of $\mathbb{R}$. This contradicts the indecomposability of $\mathbb{R}$. Therefore we have $\neg \text{WLPO}$.

Note that to prove statement (2) of Theorem 2.1 we do not use the sequential nondiscontinuity of just any function: Taking a function $f$ from $\mathbb{R}$ to $\mathbb{N}$, we only use that $f$ itself is sequentially nondiscontinuous. This means that we can strengthen the result to the following statement in BISH:

**Proposition 2.2**

Every sequentially nondiscontinuous function from $\mathbb{R}$ to $\mathbb{N}$ is constant.

This has some significance in the form of the following corollary in BISH:
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Corollary 2.3
Every continuous function from $[0, 1]$ to $\mathbb{N}$ is uniformly continuous.

Note that the similar statement

\[
\text{every continuous function from } 2^\mathbb{N} \text{ to } \mathbb{N} \text{ is uniformly continuous}
\]

is not provable in BISH. Instead it is equivalent to the fan-theoretical principle $\text{FT}_c$.

3 The Indecomposability of $\mathbb{R} \setminus \{0\}$

We now turn to the indecomposability of $\mathbb{R} \setminus \{0\}$. We show that it is equivalent to the negation of the disjunctive form of Markov’s Principle. Recall that the disjunctive form of Markov’s Principle ($\text{MP}^\lor$) is equivalent to the statement:

If $x \neq 0$, then $\neg\neg(0 < x)$ or $\neg\neg(0 > x)$ for every $x \in \mathbb{R}$.

It is not hard to prove that this is equivalent to

If $x \neq 0$, then $0 \leq x$ or $0 \geq x$ for every $x \in \mathbb{R}$.

In [8] it has been proved that LLPO implies $\text{MP}^\lor$. Because it is easy to prove that WLPO implies LLPO, which has also been remarked in [3], we get that WLPO implies $\text{MP}^\lor$, and hence that $\neg\text{MP}^\lor$ implies $\text{CONT}_{00}$ and the indecomposability of $\mathbb{R}$.

Theorem 3.1
The following statements are equivalent over BISH:

(1) $\neg\text{MP}^\lor$.
(2) Every function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{N}$ is constant.
(3) The set $\mathbb{R} \setminus \{0\}$ is indecomposable.

Proof. We will prove that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$.

1 $\Rightarrow$ 2 Let $f$ be a function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{N}$. Because of the indecomposability of $\mathbb{R}$ both the functions $f|_{(-\infty, 0)}$ and $f|_{(0, \infty)}$ are constant. Note also that for every $x \in \mathbb{R} \setminus \{0\}$ we have either $f(x) = f(1)$ or $f(x) = f(-1)$: for if $f(x) \neq f(1)$ and $f(x) \neq f(-1)$, then $x \notin (-\infty, 0) \cup (0, \infty)$, which leads to a contradiction.

Now suppose that $f(-1) \neq f(1)$. Let $x \in \mathbb{R} \setminus \{0\}$, and suppose that $f(x) = f(1)$. Suppose $x < 0$; then $f(x) = f(-1)$. This leads to a contradiction, so $x \geq 0$. Similarly, if we suppose that $f(x) = f(-1)$, we can conclude that $x \leq 0$.

Now we have a contradiction with the assumption $\neg\text{MP}^\lor$. Thus $f(-1) = f(1)$ and we conclude that $f$ is constant.

2 $\Rightarrow$ 3 Similar to the corresponding case in the proof of Theorem 2.1

3 $\Rightarrow$ 1 Suppose that $\mathbb{R} \setminus \{0\}$ is indecomposable. Suppose that $\text{MP}^\lor$. Let $y \in \mathbb{R} \setminus \{0\}$; then $y \neq 0$, so $y \leq 0$ or $y \geq 0$. This means that the subset $\{y \in \mathbb{R} \setminus \{0\} : y \leq 0\}$ is decidable, as it cannot be the case that both $y \leq 0$ and $y \geq 0$. We now have a contradiction with the indecomposability of $\mathbb{R} \setminus \{0\}$.
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References


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